Exam

Statistical Physics

Thursday January 21, 2016
9:00-12:00

Read these instructions carefully before making the exam!

- **Write your name and student number on *every* sheet.**
- ***Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing*.**
- ***Language*; your answers have to be in English.**
- **Use a *separate* sheet of paper for each problem.**
- **Use of a (graphing) calculator is allowed.**
- **This exam consists of 4 problems.**
- **The weight of the problems is: Problem 1 (P1=25 pts); Problem 2 (P2=20 pts); Problem 3 (P3=20 pts); Problem 4 (P4=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.**
- **The grade of the exam is calculated as (P1+P2+P3+P4+10)/10.**
- **For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.***

PROBLEM 1

Score: a+b+c+d+e+f=5+4+4+4+4+4=25

Consider a crystal of N independent identical particles. Each of these particles has three non-degenerate energy levels namely, $-\varepsilon$, 0 and ε. The crystal is in equilibrium with a heat bath with temperature T.

a) Calculate the partition function Z_1 for a single particle.
b) Calculate the probabilities P_{-1}, P_0 and P_1 that the energy levels $-\varepsilon$, 0 and ε of this particle are occupied.
c) Give the values of P_{-1}, P_0 and P_1 in the low-temperature limit ($T \to 0$) and the-high temperature limit ($T \to \infty$).
d) Calculate the energy E of a crystal of N particles.
e) Show that the entropy S of a crystal of N particles is given by:

$$S = N\left\{\frac{\varepsilon}{T}\frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k\ln\left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)\right\}$$

f) Find the entropy S in the low-temperature limit ($T \to 0$) and the high-temperature limit ($T \to \infty$) and discuss the results.

PROBLEM 2

Score: a+b+c+d=5+5+5+5=20

A bounded harmonic oscillator is a harmonic oscillator that has a finite number of energy levels E_n. If this number is n_0, the energy levels can be written as:

$$E_n = \varepsilon_0 + n\varepsilon, \qquad n = 0,1,2,\cdots,n_0 - 1$$

with ε_0 the ground state energy and ε the spacing between the energy levels.

a) Show that the partition function Z_1 for a single bounded harmonic oscillator is given by:

$$Z_1 = e^{-\beta\varepsilon_0}\frac{1 - e^{-n_0\beta\varepsilon}}{1 - e^{-\beta\varepsilon}}$$

b) Give the partition function Z_N for a solid consisting of N independent bounded harmonic oscillators.
c) Calculate the energy E for a solid of N independent bounded harmonic oscillators.

Now compare the bounded harmonic oscillator with the unbounded harmonic oscillator (limit $n_0 \to \infty$) .

d) For which temperatures will both oscillators show identical properties?

PROBLEM 3

Score: a+b+c+d+e=5+3+3+4+5=20

A perfect gas of N atoms is confined to a volume V. The gas has a pressure P and a temperature T. In the temperature range $[T_1, T_2]$ the heat capacity at constant volume c_V (in J molecule^{-1} K^{-1}) of the gas is constant.

a) Use thermodynamics to show that for temperatures T in this range the entropy of the gas can be written as:

$$S(N,T,V) = Nc_V \ln T + Nk \ln V + C$$

where C is a constant of integration.

b) Explain what an extensive quantity is.

c) Entropy is an extensive quantity. The expression for S derived under a) does not reflect this property of the entropy. Use the freedom in the constant of integration to derive the following expression for the entropy that does show the extensiveness of the entropy:

$$S(N,T,V) = N\left\{c_V \ln T + k \ln \frac{V}{N} + \acute{C}\right\}$$

where $\acute{C}$ is a constant that does not depend on N.

A perfect gas is compressed in an isothermal process from a volume V_1 to a volume V_2 and thereafter expanded in a reversible adiabatic process to its original volume V_1.

d) Calculate the entropy change ΔS of the perfect gas during the isothermal compression. Is ΔS positive or negative?

e) Calculate the ratio $\frac{T_a}{T_b}$ of the temperatures of the perfect gas before (T_b) and after (T_a) the reversible adiabatic expansion. Use $V_2 = 0.2V_1$ and $c_V = \frac{3}{2}\frac{R}{N_0}$ where R is the gas constant and N_0 is Avogadro's number. Assume that T_b and T_a are both in the temperature range $[T_1, T_2]$.

PROBLEM 4

Score: a+b+c+d+e+f+g=4+3+4+4+4+3+3=25

Consider a perfect gas of fermions in an enclosure with volume V that is in contact with both a heat bath and a particle reservoir. A state of the gas is described by the set of occupation numbers $n_1, n_2, \cdots n_i, \cdots$ of the single fermion states with energies $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_i \leq \cdots$, respectively.

The grand partition function $\mathcal{Z}$ for this gas of fermions is defined as:

$$\mathcal{Z} = \sum_{n_1, n_2, \cdots} e^{\beta[\mu(n_1+n_2+\cdots)-(n_1\varepsilon_1+n_2\varepsilon_2+\cdots)]}$$

And the probability of finding the gas in the state $n_1, n_2, \cdots n_i, \cdots$ is given by:

$$P(n_1, n_2, \cdots n_i, \cdots) = \frac{e^{\beta[\mu(n_1+n_2+\cdots)-(n_1\varepsilon_1+n_2\varepsilon_2+\cdots)]}}{\mathcal{Z}}$$

a) Show that this grand partition function and probability factorize as:

$$\mathcal{Z} = \prod_{i=1}^{\infty} \mathcal{Z}_i \text{ with } \mathcal{Z}_i = \sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i}$$

and

$$P(n_1, n_2, \cdots n_i, \cdots) = \prod_{i=1}^{\infty} P_i(n_i) \text{ with } P_i(n_i) = \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i}$$

b) Give the interpretation of the function $P_i(n_i)$.

c) Show that for fermions we have:

$$\mathcal{Z}_i = 1 + e^{\beta(\mu-\varepsilon_i)}$$

d) Prove that the mean occupation number $\overline{n}_i$ of the i-th single fermion state can be calculated from:

$$\overline{n}_i = \frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu}\right)_{T,V}$$

and use this expression to calculate $\overline{n}_i$.

Problem continues on other side of the page

e) Show that the total number of fermions N in a perfect gas of fermions with spin ½ is given by,

$$N = \left[\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\right]\int_0^\infty \frac{\sqrt{\varepsilon}d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

HINT: The density of states for a *spinless* particle confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$f(p)dp = \frac{V}{h^3}4\pi p^2 dp$$

f) At $T = 0$ we define the Fermi energy as $\varepsilon_F = \mu(T = 0)$. Show that at $T = 0$ the number of fermions is related to the Fermi energy as:

$$N = \left[\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\right]\frac{2}{3}(\varepsilon_F)^{\frac{3}{2}}$$

g) What is the physical interpretation of the Fermi energy?

Solutions

PROBLEM 1

a)

$$Z_1 = \sum_{n=-1,0,1} e^{-\beta E_n} = e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}$$

b)

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

c) Low temperature limit, $\beta\varepsilon \to \infty$ then

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon}} \to 1$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{1}{e^{\beta\varepsilon}} \to 0$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{0}{e^{\beta\varepsilon}} \to 0$$

High temperature limit, $\beta\varepsilon \to 0$ then

$$P_{-1} = \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{1}{1+1+1} = \frac{1}{3}$$

$$P_0 = \frac{1}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{1}{1+1+1} = \frac{1}{3}$$

$$P_1 = \frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} \to \frac{1}{1+1+1} = \frac{1}{3}$$

d)

The mean energy of 1 particle is,

$$\bar{\varepsilon} = \sum_{n=-1,0,1} P_n E_n = \frac{-\varepsilon e^{\beta\varepsilon} + 0 + \varepsilon e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} = \varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

Because the particles are independent we have,

$$E = N\bar{\varepsilon} = N\varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

Another method is through the N-particle partition function Z_N

$$Z_N = (Z_1)^N = \left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)^N$$

And use

$$E = -\frac{\partial \ln Z_N}{\partial \beta} = -N \frac{\partial}{\partial \beta}\left(\ln\left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)\right) \Rightarrow$$

$$E = -N \frac{\varepsilon e^{\beta\varepsilon} - \varepsilon e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} = N\varepsilon \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}}$$

e)
Use $F = E - TS$ and $F = -kT \ln Z_N$ to find,

$$S = \frac{E - F}{T} = \frac{E + kT \ln Z_N}{T} \Rightarrow$$

$$S = N \frac{\varepsilon}{T} \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k \ln\left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)^N \Rightarrow$$

$$S = N \left\{\frac{\varepsilon}{T} \frac{e^{-\beta\varepsilon} - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}} + k \ln\left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)\right\}$$

f)

High temperature means $\beta \to 0$ and we have

$$S \to N \frac{\varepsilon}{T} \frac{1 - 1}{1 + 1 + 1} + Nk \ln(1 + 1 + 1) = Nk \ln 3$$

At high temperature all three levels are equally likely for each particle and thus we have a contribution of $k \ln 3$ per particle to the entropy, for N particles the entropy is than $Nk \ln 3$.

At low temperatures $\beta \to \infty$ and we find:

$$S \to N\left\{\frac{\varepsilon}{T}\frac{0 - e^{\beta\varepsilon}}{e^{\beta\varepsilon} + 1 + 0} + k \ln\left(e^{\beta\varepsilon} + 1 + 0\right)\right\} \to N\left\{\frac{\varepsilon}{T}\frac{-e^{\beta\varepsilon}}{e^{\beta\varepsilon}} + k \ln\left(e^{\beta\varepsilon}\right)\right\}$$
$$= N\left\{-\frac{\varepsilon}{T} + k\beta\varepsilon\right\} = N\left\{-\frac{\varepsilon}{T} + \frac{k\varepsilon}{kT}\right\} = 0$$

According to the third law a system with non-degenerate ground state has zero entropy. In this case all particles are in the $-\varepsilon$ state. There is only 1 such configuration, thus $S = k \ln 1 = 0$.

PROBLEM 2

a)

$$Z_1 = \sum_{n=0}^{n_0-1} e^{-\beta E_n} = \sum_{n=0}^{n_0-1} e^{-\beta(\varepsilon_0+n\varepsilon)} = e^{-\beta\varepsilon_0} \sum_{n=0}^{n_0-1} e^{-\beta n\varepsilon} = e^{-\beta\varepsilon_0} \sum_{n=0}^{n_0-1} \left(e^{-\beta\varepsilon}\right)^n$$

$$= e^{-\beta\varepsilon_0} \sum_{n=0}^{n_0-1} r^n$$

with $r = e^{-\beta\varepsilon}$

If we consider the sum

$$S = \sum_{n=0}^{n_0-1} r^n = 1 + r + r^2 + \cdots + r^{n_0-1}$$

Then we have

$$rS = r + r + r^2 + \cdots + r^{n_0}$$

Subtracting both equations results in:

$$S - rS = 1 - r^{n_0} \Rightarrow S = \frac{1 - r^{n_0}}{1 - r}$$

And using this in the earlier expressions we find,

$$Z_1 = e^{-\beta\varepsilon_0} \frac{1 - e^{-n_0\beta\varepsilon}}{1 - e^{-\beta\varepsilon}}$$

b)
Because the oscillators are independent we have,

$$Z_N = (Z_1)^N = e^{-N\beta\varepsilon_0} \left(\frac{1 - e^{-n_0\beta\varepsilon}}{1 - e^{-\beta\varepsilon}}\right)^N$$

c)

$$E = -\frac{\partial \ln Z_N}{\partial\beta} = -\frac{\partial}{\partial\beta}\left(-N\beta\varepsilon_0 + N\ln\left(1 - e^{-n_0\beta\varepsilon}\right) - N\ln\left(1 - e^{-\beta\varepsilon}\right)\right) \Rightarrow$$

$$E = -\frac{\partial \ln Z_N}{\partial\beta} = N\varepsilon_0 - \frac{N n_0 \varepsilon e^{-n_0\beta\varepsilon}}{1 - e^{-n_0\beta\varepsilon}} + \frac{N\varepsilon e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}} \Rightarrow$$

$$E = -\frac{\partial \ln Z_N}{\partial\beta} = N\left(\varepsilon_0 - \frac{n_0\varepsilon}{e^{n_0\beta\varepsilon} - 1} + \frac{\varepsilon}{e^{\beta\varepsilon} - 1}\right)$$

d)
The partition function Z_1^∞ for a single unbounded oscillator is (take the limit $n_0 \to \infty$) :

$$Z_1^\infty = e^{-\beta\varepsilon_0} \frac{1}{1 - e^{-\beta\varepsilon}}$$

If both partition functions Z_1^∞ and Z_1 are approximately equal, the physics (which emerges from the partition function) is approximately equal.

Thus when,

$$e^{-n_0\beta\varepsilon} \approx 0 \Rightarrow \frac{n_0\varepsilon}{kT} \gg 0 \Rightarrow kT \ll n_0\varepsilon$$

For these temperatures only the energy levels below E_{n_0} are occupied and there is no difference between both oscillators. In case $kT \gg n_0\varepsilon$ the physics of both oscillators is different. The above is sufficient answer for the full credits.

If we consider the energy in this limit we have for the bounded oscillator that all energy levels become equally probable, thus the total energy is:

$$E = N \frac{\varepsilon_0 + (\varepsilon_0 + \varepsilon) + (\varepsilon_0 + 2\varepsilon) + \cdots + (\varepsilon_0 + (n_0 - 1)\varepsilon)}{n_0} \Rightarrow$$

$$E = N\varepsilon_0 + \frac{\varepsilon + 2\varepsilon + \cdots + (n_0 - 1)\varepsilon}{n_0} = N\varepsilon_0 + \frac{\frac{1}{2}(n_0 - 1)(n_0)\varepsilon}{n_0} = N\left(\varepsilon_0 + \frac{\varepsilon}{2}[n_0 - 1]\right)$$

Another way to arrive at this result is:

$$E = N\left(\varepsilon_0 - \frac{n_0\varepsilon}{e^{n_0\beta\varepsilon} - 1} + \frac{\varepsilon}{e^{\beta\varepsilon} - 1}\right) = N\left(\varepsilon_0 - n_0\varepsilon\left[\frac{1}{e^{n_0\beta\varepsilon} - 1} - \frac{1}{n_0(e^{\beta\varepsilon} - 1)}\right]\right) \Rightarrow$$

$$E \approx N\left(\varepsilon_0 - n_0\varepsilon\left[\frac{1}{1 + n_0\beta\varepsilon + \frac{1}{2}(n_0\beta\varepsilon)^2 - 1} - \frac{1}{n_0\left(1 + \beta\varepsilon + \frac{1}{2}(\beta\varepsilon)^2 - 1\right)}\right]\right) \Rightarrow$$

$$E \approx N\left(\varepsilon_0 - \frac{n_0\varepsilon}{n_0\beta\varepsilon}\left[\frac{1}{1 + \frac{1}{2}n_0\beta\varepsilon} - \frac{1}{1 + \frac{1}{2}\beta\varepsilon}\right]\right) \Rightarrow$$

$$E \approx N\left(\varepsilon_0 - \frac{1}{\beta}\left[1 - \frac{1}{2}n_0\beta\varepsilon - 1 + \frac{1}{2}\beta\varepsilon\right]\right) \Rightarrow$$

$$E \approx N\left(\varepsilon_0 + \frac{\varepsilon}{2}[n_0 - 1]\right)$$

For the unbounded oscillator we have when $kT \gg \varepsilon$,

$$E = N\left(\varepsilon_0 + \frac{\varepsilon}{e^{\beta\varepsilon} - 1}\right) \approx N\left(\varepsilon_0 + \varepsilon\left[\frac{1}{\beta\varepsilon}\right]\right) = N\left(\varepsilon_0 + \varepsilon\left[\frac{kT}{\varepsilon}\right]\right)$$

This is unbounded, no maximum energy.

PROBLEM 3

Start with:

$$dE = TdS - PdV \Rightarrow dS = \frac{1}{T}dE + \frac{P}{T}dV \Rightarrow$$

Use V and T as the independent variables,

$$dS = \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V dT + \frac{1}{T}\left(\frac{\partial E}{\partial V}\right)_T dV + \frac{P}{T}dV = Nc_V\frac{dT}{T} + 0 + \frac{P}{T}dV = Nc_V\frac{dT}{T} + Nk\frac{dV}{V}$$

In the second step we used: $\left(\frac{\partial E}{\partial T}\right)_V = Nc_V$ by definition of the heat capacity and $\left(\frac{\partial E}{\partial V}\right)_T = 0$ because for a perfect gas $E = E(T)$. In the second step the equation of state of a perfect gas was used: $PV = NkT$.

Integrating and use that c_V does not depend on temperature,

$$S = \int Nc_V\frac{dT}{T} + Nk\frac{dV}{V} = Nc_V\int\frac{dT}{T} + Nk\int\frac{dV}{V} = Nc_V\ln T + Nk\ln V + C$$

The integration constant C does not depend on T and V but may depend on N.

b)
An extensive quantity is a quantity that is proportional to the size of the system.

c)
Choose $C = -Nk\ln N + N\acute{C}$

d)

$$\Delta S = S_{after} - S_{before} = S(N,T,V_2) - S(N,T,V_1) = Nk\ln\frac{V_2}{V_1}$$

As the gas is compressed we have $\frac{V_2}{V_1} < 1$ and thus $\Delta S < 0$.

e)
In the reversible adiabatic process no heat is exchanged and thus $dS = \frac{\bar{d}Q}{T} = 0 \Rightarrow \Delta S = 0$.

$$\Delta S = S_{after} - S_{before} = S(N,T_a,V_1) - S(N,T_b,V_2) = Nc_V\ln\frac{T_a}{T_b} + Nk\ln\frac{V_1}{V_2}$$

Consequently;

$$N c_V \ln\frac{T_a}{T_b} + Nk\ln\frac{V_1}{V_2} = 0 \Rightarrow \ln\frac{T_a}{T_b} = -\frac{k}{c_V}\ln\frac{V_1}{V_2} = -\frac{N_0 k}{N_0 c_V}\ln\frac{V_1}{V_2} = -\frac{R}{\frac{3}{2}R}\ln\frac{V_1}{V_2} \Rightarrow$$

$$\frac{T_a}{T_b} = \left(\frac{V_1}{V_2}\right)^{-\frac{2}{3}} = \left(\frac{V_2}{V_1}\right)^{\frac{2}{3}} = (0.2)^{\frac{2}{3}} = 0.34$$

PROBLEM 4

a)

$$\mathcal{Z} = \sum_{n_1, n_2, \cdots} e^{\beta[\mu(n_1+n_2+\cdots)-(n_1\varepsilon_1+n_2\varepsilon_2+\cdots)]} = \sum_{n_1, n_2, \cdots} e^{\beta(\mu-\varepsilon_1)n_1+\beta(\mu-\varepsilon_2)n_2+\cdots} \Rightarrow$$

$$\mathcal{Z} = \sum_{n_1, n_2, \cdots} e^{\beta(\mu-\varepsilon_1)n_1} e^{\beta(\mu-\varepsilon_2)n_2} \times \cdots = \sum_{n_1} e^{\beta(\mu-\varepsilon_1)n_1} \sum_{n_2} e^{\beta(\mu-\varepsilon_2)n_2} \times \cdots \Rightarrow$$

The equality above holds because for each n_i the factor $e^{\beta(\mu-\varepsilon_i)n_i}$ is a constant for all the sums over n_j, with $j \neq i$.

$$\mathcal{Z} = \prod_{i=1}^{\infty} \sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i} = \prod_{i=1}^{\infty} \mathcal{Z}_i$$

Start with

$$P(n_1, n_2, \cdots n_i, \cdots) = \frac{e^{\beta[\mu(n_1+n_2+\cdots)-(n_1\varepsilon_1+n_2\varepsilon_2+\cdots)]}}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \cdots n_i, \cdots) = \frac{e^{\beta(\mu-\varepsilon_1)n_1+\beta(\mu-\varepsilon_2)n_2+\cdots}}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \cdots n_i, \cdots) = \frac{e^{\beta(\mu-\varepsilon_1)n_1} e^{\beta(\mu-\varepsilon_2)n_2} \times \cdots}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \cdots n_i, \cdots) = \frac{\prod_{i=1}^{\infty} e^{\beta(\mu-\varepsilon_i)n_i}}{\prod_{i=1}^{\infty} \mathcal{Z}_i} \Rightarrow$$

$$P(n_1, n_2, \cdots n_i, \cdots) = \prod_{i=1}^{\infty} P_i(n_i) \text{ with } P_i(n_i) = \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i}$$

This means that the probability to find n_i fermions in the i-th single energy state is independent of the occupancies of all other single energy states.

b)
The function $P_i(n_i)$ is the probability of finding n_i fermions in the *i*-th single fermion state.

c)
There can only be zero or one fermion in each single fermion state ($n_i = 0,1$) thus,

$$\mathcal{Z}_i = \sum_{n_i=0,1} e^{\beta(\mu-\varepsilon_i)n_i} = e^0 + e^{\beta(\mu-\varepsilon_i)} = 1 + e^{\beta(\mu-\varepsilon_i)}$$

d)
The mean occupation number is defined as:

$$\overline{n}_i = \sum_{n_i} n_i \, P_i(n_i)$$

Performing the differentiation:

$$\frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu}\right)_{T,V} = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln\left(\sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i}\right) = \frac{\sum_{n_i} n_i e^{\beta(\mu-\varepsilon_i)n_i}}{\sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i}} = \sum_{n_i} n_i \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i} \Rightarrow$$

$$\frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu}\right)_{T,V} = \sum_{n_i} n_i \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i} = \sum_{n_i} n_i P_i(n_i) = \overline{n}_i$$

We now can calculate $\overline{n}_i$ as:

$$\overline{n}_i = \frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu}\right)_{T,V} = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln\left(1 + e^{\beta(\mu-\varepsilon_i)}\right) = \frac{e^{\beta(\mu-\varepsilon_i)}}{1 + e^{\beta(\mu-\varepsilon_i)}} = \frac{1}{e^{\beta(\varepsilon_i-\mu)} + 1}$$

e)

$$N = \int_0^\infty f(\varepsilon) n(\varepsilon) d\varepsilon$$

In this $f(\varepsilon)d\varepsilon$ is the density of states and $n(\varepsilon)$ is the mean occupation number.
The density of states follows from the hint and converting momentum to energy (using $= \frac{p^2}{2m}$) as the variable and adding a factor of two for the two spin states for a fermion with spin ½ . Substitute

$$p^2 = 2m\varepsilon \text{ and } 2pdp = 2(2m\varepsilon)^{\frac{1}{2}}dp = 2md\varepsilon \Rightarrow dp = \frac{(2m)^{\frac{1}{2}}}{2\sqrt{\varepsilon}}d\varepsilon$$

in

$$f(p)dp = 2\frac{V}{h^3}4\pi p^2 dp$$

to find,

$$f(\varepsilon)d\varepsilon = 2\frac{V}{h^3}4\pi 2m\varepsilon\frac{(2m)^{\frac{1}{2}}}{2\sqrt{\varepsilon}}d\varepsilon = \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\sqrt{\varepsilon}d\varepsilon$$

The mean occupation number is (from d):

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$N = \int_0^\infty \frac{\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)} + 1}d\varepsilon = \left[\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\right]\int_0^\infty \frac{\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)} + 1}d\varepsilon$$

f)
At $T = 0$ $n(\varepsilon) = 1$ if $\varepsilon < \varepsilon_F$ and $n(\varepsilon) = 0$ if $\varepsilon > \varepsilon_F$ thus

$$N = \left[\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\right]\int_0^{\varepsilon_F} \sqrt{\varepsilon}d\varepsilon = \left[\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\right]\frac{2}{3}(\varepsilon_F)^{\frac{3}{2}}$$

g)
At $T = 0$ the system is in its state of minimum energy. Because of Pauli exclusion we have one fermion in each state. All the N lowest particle states are filled. The Fermi energy is the energy of the level with the highest energy that is filled at $T = 0$.